I. 17. Classical Periodic Orbits in Reflection-Asymmetric Deformed Cavity –Fission Model–

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Introduction

The semiclassical origins of shell structures widely observed in finite fermion systems are not yet fully understood and needs to be explored further in the language of semiclassical periodic orbit theory\textsuperscript{1−3}. Recently, the significant correlation between the bifurcation of periodic orbits and the appearance of quantum shell at stationary nuclear deformations was suggested by several authors\textsuperscript{4−6}. In our last report\textsuperscript{6}, the investigation of the superdeformed shell structures in the elliptic billiard was carried out and led to both quantitative and qualitative evidence for the importance of the orbital bifurcation in the emergence of shell structure. Since all previous works were, however, based on the use of highly-symmetric cavity models, I have investigated the effects due to the removal of such higher degeneracy and determined the role of the orbital bifurcation in general axially-symmetric, but non-integrable cavity: fission model.

Methods and Results

Nuclear Shape

In the present report, I employed the axially-symmetric fission model based on Ref. 7, and the shape of the nuclear surface is defined in the dimensionless coordinates (u, v) by an equation

\[ \pi(\beta,u,v)=v^2-1/2(A+Bu^2+\alpha u), \]  \hspace{1cm} (1)

where \( \beta \) is a set of deformation parameters, and \( u \) and \( v \) are related to the ordinary polar coordinates (\( \rho, z \)) by

\[ z=cr_0u, \rho=cr_0v. \]  \hspace{1cm} (2)

Here, \( c \) is a dimensionless elongation parameter, and \( r_0 \) is the radius of the sphere.

The parameter \( \alpha \) in Eq. 1 represents the asymmetry of the nuclear surface along \( z \) direction, and a family of symmetric shapes can be produced for \( \alpha=0 \). In addition, the coefficients \( A \) and \( B \) in Eq. 1 are expressed in terms of a deformation parameter \( c \) and a necking parameter \( h \) as

\[ B=2h+(c-1)/2, A=(1/c^2)-B/5. \]  \hspace{1cm} (3)

Figure 1 demonstrates a set of shapes in the \( c-h \) plane for some fixed values of \( \alpha \).
Single-particle energies of a particle moving freely inside of the considered cavity can be obtained by solving Schrödinger equation under Dirichlet boundary condition with the spherical-wave decomposition method.\(^{4,5}\)

### Bifurcation of Periodic Orbits

As is well known for the case of the spheroidal cavity, i.e., removal of spherical symmetry, the linear orbits along diameter in the spherical cavity bifurcate into those along the major axis and along the minor axis, while the planar orbits at the spherical limit bifurcate into those in the meridian and equatorial planes. Further shape deformation also leads to the bifurcations of linear and planar orbits in the equatorial plane into hyperbolic orbits in the meridian plane and three-dimensional orbits (see Fig. 2a). According to Ref. 8, such three-dimensional orbits for axially symmetric cavity can be created when the equatorial orbit labeled by \((p, t)\) with \(p\) and \(t\) being the numbers of vibration and rotation meets the bifurcation condition, see also Fig. 2b,

\[
\frac{R_2}{R_1} = \sin^2\left(\frac{\pi t}{p}\right)
\]

where \(R_1\) and \(R_2\) represent the curvatures of the cavity at the edge of equatorial plane, and \(q\) is an integer. Figure 4 shows a set of bifurcation lines, i.e., constant curvature ratio given by Eq. (4), as functions of both \(c\) and \(h\) at \(\alpha=0\).

Based on our recent findings that the orbital bifurcation satisfying Eq. (4) is responsible for the emergence of shell structure, one must clarify whether the important role of the orbital bifurcation in shell structure is even present in non-integrable cavities. In the following sections, the shell structures are compared along the constant curvature ratio where all equatorial orbits complete their bifurcation process for the creation of the three-dimensional orbits, and six points labeled by \(a\) through \(f\) are chosen for a comparison, as shown in Fig. 4, where a point \(f\) corresponds to a superdeformed spheroidal shape.

### Shell Structure and Fourier Spectra

The shell-structure energy \(\delta E\) can be defined as the difference between the sum of single-particle energies filling \(N\) states from the bottom of the well and the Strutinsky averaged energies, i.e.,

\[
\delta E = 2 \sum_{i=1}^{N} \epsilon_i - \bar{E}, \quad \bar{E} = 2 \int d\epsilon' \bar{g}(\epsilon'),
\]

with the Fermi energy \(\bar{\epsilon}_F\) satisfying

\[
N = 2 \int d\epsilon' \bar{g}(\epsilon').
\]

In Fig. 4, the shell-structure energy for the case of the spheroidal cavity is shown for mere reference only. The emergence of the prominent magic gaps seen at the superdeformed shape can be regarded as the result of the orbital bifurcations as demonstrated by previous studies.

Figure 5 illustrates the oscillating pattern of the shell-structure energy \(\delta E\) as
functions of both $c$ and $h$ for a fixed particle number $N$, and it is compared with the constant orbital bifurcation lines $R/R_3$ and the elongation of the surface, $\chi$, defined by

$$\chi = \int \rho \frac{dV}{\int \rho dV}.$$  \hspace{1cm} (7)

It is clear from the figure that for small elongation the shell valley follows the constant elongation points, see Fig. 5a, whereas the role of the bifurcated equatorial orbits increases for larger values of $\chi$, see Fig. 5b.

In terms of semiclassical trace formula, the level density $g_{sc}(\varepsilon)$ can be written as

$$g_{sc}(\varepsilon) = \tilde{g}(\varepsilon) + \sum_b A_b(k) \cos \left( kL_b - \frac{\pi}{2} \mu_b \right),$$  \hspace{1cm} (8)

where $\tilde{g}(\varepsilon)$ denotes the smooth part corresponding to the contribution of zero-length orbit, $A_b$ the amplitude, and $\mu_b$ the Maslov phase.

Since the oscillating part of the semiclassical level density in Eq. 7 represents the Fourier sum, the classical-quantum correspondence on shell structure can be obtained from the Fourier transform $F(L)$ of the quantum level density $g(\varepsilon)$ with respect to the wave number, which may be regarded as 'length spectrum' exhibiting peaks at lengths of individual periodic orbits.

Figure 6 illustrates the deformation dependence of the Fourier peaks for six previously chosen shapes. The nuclear shape corresponding to a point $a$ results in strong shell fluctuation, and its Fourier spectrum indicates that the main contribution comes from the meridian planar orbits of lengths roughly ranging from 5 to 7, while the peaks related to the three-dimensional orbits bifurcated from the equatorial orbits, labeled by $\alpha, \beta, \gamma$ in the figure, are less pronounced in the moderately-elongated region. When the system starts to elongate further, the shell oscillations due the meridian and three-dimensional orbits contribute destructively, leading to flattening of the shell structure, see Figs. 6b-e. As the nuclear shape approaches to the highly-symmetric superdeformed spheroidal one as in Fig. 6f, the origin of the prominent shell structure can be understood in terms of the constructive interference among the three-dimensional orbits originated from the equatorial planar orbits.

Conclusions

The role of classical periodic orbits in the emergence of shell structure for single-particle motions in axially-symmetric fission cavity was identified. In particular, the importance of orbital bifurcation as indicated by previous investigations for highly symmetric (elliptic and spheroidal) cavities was carefully analyzed.

It was shown that the contributions from the bifurcated orbits originated from the equatorial planar orbits play a significant role in the regions around superdeformed spheroidal shape, while its enhancement becomes less pronounced for less-elongated shapes, indicating the importance of the orbital bifurcation in the emergence of prominent quantum shells in highly-deformed fermion systems.

It remains as a future challenge to investigate the semiclassical origin of shell formation in finite fermion systems for finite-depth potentials, and the Green’s function approach$^8$ may be applied to calculate the continuum level density in such systems.
References

Fig. 1. Shapes in the $c$-$h$ parameterization at $\alpha=0$ (solid lines) and $\alpha=0.3$ (dotted lines).

Fig. 2. (a) Schematic drawings of the orbital bifurcation of the equatorial triangular orbit (6,2) as an example. (b) Description of curvatures $R_1$ and $R_2$ given in Eq.4.

Fig. 3. Constant bifurcation lines defined by $R_1/R_2$ in Eq. 4 are plotted as functions of two deformation parameters $c$ and $h$ at $\alpha=0$ for four types of equatorial planar orbits and the corresponding orbits are also shown schematically (right figures).
Fig. 4. Shell-structure energy $\delta E$ for the spheroidal cavity as functions of axis ratio $\eta$ and occupation particle number $N$. A superdeformed spheroidal shape at $\eta=2$ corresponds to a point $f$ in Fig. 3, and its magic numbers are indicated.

Fig. 5. Shell-structure energy $\delta E$ for the axially symmetric cavities parameterized by $c$ and $h$ at $\alpha=0$ for the occupation particle number $N=20$. Solid lines in (a) and (b) represent the constant elongation and bifurcation points, respectively.
Fig. 6. (a) Shell-structure energy $\delta E$ calculated for six deformed shapes specified in Fig. 3 as a function of $N^{1/3}$, $N$ being the occupation particle number, and (b) their Fourier spectra are shown. The peaks labeled by $\alpha, \beta, \gamma$ correspond to the three-dimensional orbits bifurcated from the equatorial (4,2), (5,2), (6,2) orbits, respectively.